

Tunable Lyapunov exponent in inverse magnetic billiards

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The stability properties of the classical trajectories of charged particles are investigated in a two dimensional stadium-shaped inverse magnetic domain, where the magnetic field is zero inside the stadium domain and constant outside. In the case of infinite magnetic field the dynamics of the system is the same as in the Bunimovich billiard, i.e., ergodic and mixing. However, for weaker magnetic fields the phase space becomes mixed and the chaotic part gradually shrinks. The numerical measurements of the Lyapunov exponent (performed with a novel method) and the integrable/chaotic phase space volume ratio show that both quantities can be smoothly tuned by varying the external magnetic field. A possible experimental realization of the arrangement is also discussed.

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In the past two decades, developments in nanotechnology have made it possible to electrostatically confine a two-dimensional electron gas (2DEG) in high mobility heterostructures [1]. In these systems the dynamics of the electrons is dominated by ballistic motion. Recently, a new perspective of the research of semiconductor systems has been emerged by the application of spatially inhomogeneous magnetic fields. The inhomogeneity of the magnetic field can be realized experimentally either by varying the topography of the electron gas [2], or using ferromagnetic materials [3], or depositing a superconductor on top of the 2DEG [4]. Numerous theoretical works also show the increasing interest in the study of electron motion in inhomogeneous magnetic field [5].

The aim of this Letter is to present a novel, experimentally realizable ballistic 2DEG system which exhibits a crossover between a well known, ergodic and mixing billiard system (the Bunimovich stadium billiard [6]), and a pathological integrable system, as the applied magnetic field is changed. We suppose that the system is in the ballistic regime, like in many other works (see e.g. [1, 7]), and our treatment is purely classical. Two characteristic quantities of the dynamics of this so-called *inverse magnetic billiard* are calculated numerically as a function of the external magnetic field β : the Lyapunov exponent $\lambda(\beta)$ (of the dominating chaotic component), and the integrable/chaotic phase space volume ratio $\varrho(\beta)$. The obtained numerical results show that both quantities are smooth functions of the magnetic field which means that the global dynamics of the system passes continuously from the integrable ($\beta = 0$) to the fully chaotic case ($\beta = \infty$). As we shall see below, there is also a clearly visible correlated dependence between the variation of the quantities $\lambda(\beta)$ and $\varrho(\beta)$. These results, i.e., the fact that *the degree of chaoticity can smoothly be tuned* by the external magnetic field, may motivate the experimental realization and study of our presently proposed system. Kosztin et al. have made similar investigations and observations in Andreev billiard systems [8].

More specifically, the system we suggest is a 2DEG in

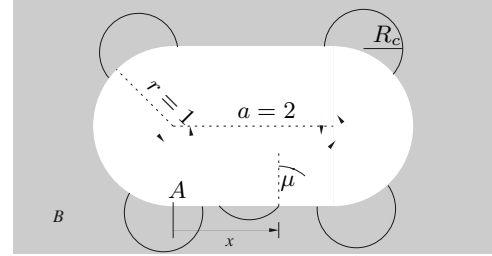


FIG. 1: The trajectories of a charged particle in the inverse magnetic billiard. The cyclotron radius is $R_c = 1/\beta = 1/2$, in dimensionless units.

an inhomogeneous magnetic field applied perpendicularly to the system. The magnetic field is considered to be zero inside a stadium-shaped region and constant β outside. This arrangement can be realized experimentally by depositing a stadium-shaped superconductor patch on the top of a 2DEG and applying an external homogeneous magnetic field perpendicular to this structure. The magnetic field is excluded from the region covered by the superconductor, due to the Meissner effect. A part of a typical classical trajectory is depicted in Fig 1, for an intermediate value of the magnetic field $\beta = 2$. The trajectories in the configuration space are straight segments inside the stadium, and circular arcs of cyclotron radius $R_c = \frac{1}{\beta}$ out of this domain. (We assume, for simplicity, that the particle has unit mass, charge and speed.) At the boundary of the domain the two pieces of the trajectory join tangentially. As the magnetic field tends to infinity, $\beta \rightarrow \infty$, the electrons spend less and less time outside the stadium, and it is also easy to see that in the limiting case their motion is described by an elastic reflection from the wall. For this reason we call our system *inverse magnetic billiard*, although in the case of finite field no real scatterings take place at the boundaries.

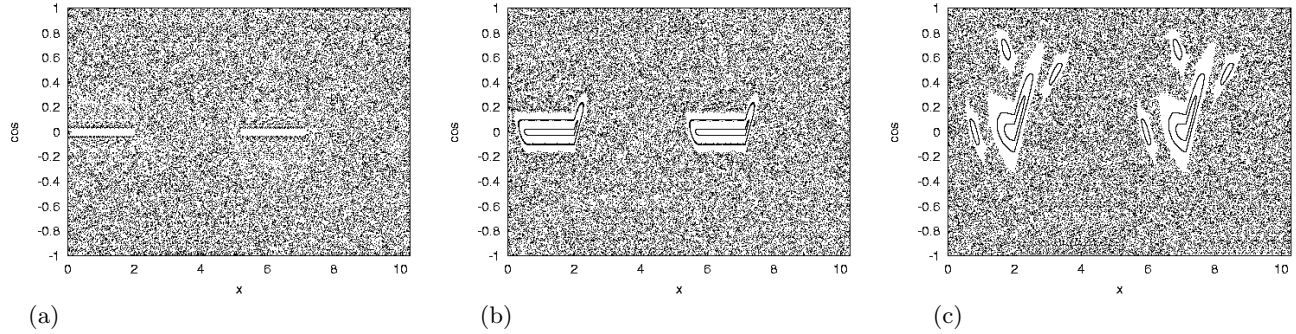


FIG. 2: The Poincaré section of the phase space. The points in the dominating chaotic region were obtained by 50000 iterations of a single trajectory, while for depicting the islands corresponding to the integrable regions, a few different initial conditions were used. The values of the cyclotron radii are $R_c = 0.05$, $R_c = 0.3$, $R_c = 1$, respectively.

According to the result of Bunimovich [6], the stadium-shaped inverse magnetic billiard system is ergodic and mixing in the $\beta = \infty$ case, but as the magnetic field is decreased, the dynamics becomes partially integrable and gradually more and more phase space volume is occupied by the KAM tori (mixed phase space). This phenomenon can clearly be observed on the Poincaré sections (see Fig. 2) made for different magnetic field values. The individual points in the Poincaré sections are plotted each time the particle enters the zero magnetic field region and crosses the boundary of the stadium. The x coordinate of the points ($0 \leq x < 4 + 2\pi$) gives the position of the crossing, measured in anti-clockwise direction from the point A along the perimeter of the stadium, while the y coordinate of the points ($-1 \leq y \leq 1$) denotes the sine of the angle μ representing the direction of the trajectory, relative to the normal of the boundary (see Fig. 1). It is well-known that in this parameter space the Poincaré map is area preserving [9].

It is evident from Fig. 2 that for high magnetic fields the system is (almost) completely chaotic but with decreasing magnetic field, the volume of the integrable regions gradually increases. As we have seen before, for $\beta = \infty$ the system is identical to the Bunimovich billiard, however, in the $\beta \rightarrow 0$ limit the system becomes pathological in the sense that the cyclotron radius tends to infinity, so the electron returns to the stadium domain after longer and longer time intervals.

In order to quantitatively characterize this change of the phase space portrait we have numerically investigated the integrable/chaotic phase space volume ratio ϱ as a function of the cyclotron radius $R_c = 1/\beta$ (i.e., the inverse magnetic field), and the results are shown in Fig. 3. The function $\varrho(R_c)$, measured by the box-counting method with a grid of 250×250 rectangular sites, is smooth, and its behavior is characteristically different for higher and lower magnetic fields. For cyclotron radii less than $R_1 \approx 0.01$ (i.e., for magnetic fields larger than $\beta_1 \approx 100$) the system is dominantly chaotic, the area of the integrable phase space regions is practically negligible (see also Fig. 2.a). For cyclotron radii larger

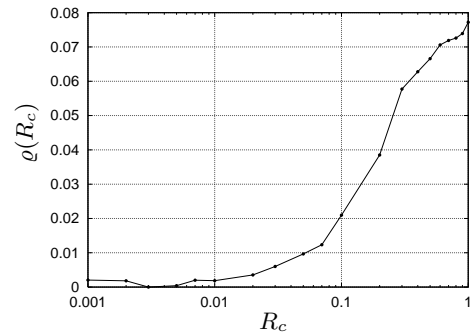


FIG. 3: The integrable/chaotic phase space volume ratio as a function of $R_c = 1/\beta$.

than $R_2 \approx 0.3$, however, the chaotic part increases on the Poincaré section (see also Fig. 2.c). Between these two extremities, i.e., for cyclotron radii comparable to the characteristic size of the billiard, the phase space of the system is definitely mixed (Fig. 2.b) with integrable islands of considerable area.

Although the volume of the chaotic bands inside the integrable islands (ignored in our treatment) is nonzero in principle, the numerical simulations demonstrate (see Fig 2) that their contribution to the chaotic phase space volume is negligible for this system.

Since the positivity of the Lyapunov exponent $\lambda(R_c)$ is one of the most characteristic features of chaotic systems, we have also numerically computed $\lambda(R_c)$ of the dominating chaotic component as a function of the cyclotron radius R_c (see Fig. 4).

The obtained function $\lambda(R_c)$ is again smooth, as $\varrho(R_c)$. It is also clearly visible that the numerical value of the Lyapunov exponent strongly correlates with the integrable phase space ratio $\varrho(R_c)$ measured previously. For weak magnetic fields (if $\beta \lesssim \beta_2 \approx 2$) the Lyapunov exponent is also small, but as the magnetic field grows, the value of λ increases, too, and for strong fields (if $\beta \gtrsim \beta_1 \approx 100$) it saturates at the value $\lambda_\infty \approx 0.43$, which agrees well with the Lyapunov exponent of the ordinary

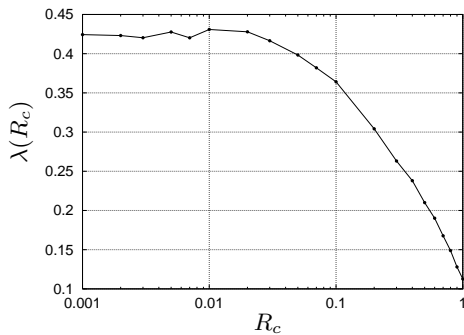


FIG. 4: Lyapunov exponent as a function of $R_c = 1/\beta$.

Bunimovich billiard [10].

In order to measure the Lyapunov exponent, we have investigated the infinitesimal variations of the trajectories with the method of Jacobi fields, which was originally developed for the stability analysis of the geodesic flow on curved Riemannian manifolds [11]. The method has successfully been applied to magnetic billiard systems on planar [12] as well as curved surfaces [13, 14]. The main idea of the method is to study the evolution of the so-called Jacobi fields along a particular trajectory in the configuration space, which describe the infinitesimal variations of the trajectory. This technique is essentially the same as the method using the tangent map [9], but our approach is more transparent. The basic technical novelty is that in our investigations the coordinates describing the infinitesimal variations are chosen in a more natural way: they are related to the unvaried trajectory itself, and not to the somewhat artificial parameters of the space of the Poincaré section. As a result, the stability matrices (i.e., the tangent maps) have a much simpler form.

In more details, let $\gamma_0(t)$ denote the trajectory in the configuration space \mathcal{M} , whose stability properties we intend to investigate, and let $\gamma_\varepsilon(t)$ be a one-parameter family of varied trajectories around the unvaried one γ_0 , i.e., for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$ the curve γ_ε is a real trajectory in the configuration space, $\gamma_{\varepsilon=0} = \gamma_0$, and the map $\gamma : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R} \rightarrow \mathcal{M}$, $(\varepsilon, t) \mapsto \gamma_\varepsilon(t)$ is everywhere continuous, and piecewise smooth. (It is not smooth at the boundary of the billiard.) The *Jacobi field* or *infinitesimal variation vector field* V_{γ_0} corresponding to the variation γ_ε is the partial derivative $V_{\gamma_0}(t) = \left. \frac{\partial \gamma_\varepsilon(t)}{\partial \varepsilon} \right|_{\varepsilon=0}$.

It can be shown that the Jacobi fields $V_{\gamma_0}(t)$ satisfy certain second order differential equation, called *Jacobi equation*; it is due to the fact that the varied curves γ_ε are also real trajectories [11, 13]. In two dimensional billiard systems we found it convenient to fix the base vectors $\{\dot{\gamma}_0(t), \dot{\gamma}_0^\perp(t)\}$ of the coordinate system to the investigated trajectory $\gamma_0(t)$, in such a way that $\dot{\gamma}_0(t)$ is the (unit) vector tangential to the trajectory at the time instant t , and $\dot{\gamma}_0^\perp(t)$ is obtained from $\dot{\gamma}_0(t)$ by a rotation through $+90^\circ$. In this basis the Jacobi field is

written as $V_{\gamma_0}(t) = \xi(t) \cdot \dot{\gamma}_0(t) + \eta(t) \cdot \dot{\gamma}_0^\perp(t)$, and for characterizing a given infinitesimal variation the initial conditions $\xi(t_0)$, $\eta(t_0)$, $\dot{\xi}(t_0)$ and $\dot{\eta}(t_0)$ have to be given. (The real functions ξ and η are the coordinates of the Jacobi field V_{γ_0} .)

The number of these initial data can further be reduced by two, if we notice that *i)* the longitudinal variations $\xi(t)$ as well as *ii)* the variations altering the speed (i.e., for which $\dot{\xi} \neq 0$) are irrelevant in the present investigation, and they decouple from the other coordinates, so they can be disregarded. (In the case *i)* the Jacobi field is tangential to the unvaried trajectory γ_0 , thus the varied curves are just time-shifts of the original one, while *ii)* means that we restrict the attention to a constant energy shell of the phase space, as it is usual in Hamiltonian systems.)

In planar billiard systems it is an elementary geometric problem to find the solutions of the Jacobi equation in terms of the transverse coordinates $\eta(t)$ and $\dot{\eta}(t)$ (see e.g. [12]). Generally, the solution is given by a linear transformation $\begin{bmatrix} \eta' \\ \dot{\eta}' \end{bmatrix} = \mathbf{L} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix}$, where the matrix \mathbf{L} has the following special forms for the straight flight in zero magnetic field (\mathbf{P}), for the curved flight in nonzero magnetic field (\mathbf{E}) and for the boundary transition (\mathbf{T}) with magnetic field change $\Delta\beta$, respectively:

$$\mathbf{P}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad (1a)$$

$$\mathbf{E}(t, \beta) = \begin{bmatrix} \cos(\beta t) & \frac{1}{\beta} \sin(\beta t) \\ -\beta \sin(\beta t) & \cos(\beta t) \end{bmatrix}, \quad (1b)$$

$$\mathbf{T}(\Delta\beta, \mu) = \begin{bmatrix} 1 & 0 \\ \Delta\beta \tan \mu & 1 \end{bmatrix}. \quad (1c)$$

Here t is the time of flight (so βt is the angle of flight), β denotes the magnetic field and μ is the angle of incidence at the boundary, measured in the way shown in Fig 1. It is worth noticing that all the three types of matrices are one-parameter subgroups of $SL(2, \mathbb{R})$, i.e., of the group of two by two real matrices with unit determinant. The matrices \mathbf{P} and \mathbf{T} are parabolic, while the transformations \mathbf{E} are elliptic.

For investigating the long time stability of a given trajectory γ_0 the eigenvalues (or the trace) of the product matrix

$$\dots (\mathbf{T}'_3 \mathbf{E}_3 \mathbf{T}_3 \mathbf{P}_3) (\mathbf{T}'_2 \mathbf{E}_2 \mathbf{T}_2 \mathbf{P}_2) (\mathbf{T}'_1 \mathbf{E}_1 \mathbf{T}_1 \mathbf{P}_1) \quad (2)$$

have to be calculated, where the individual matrices in the expression describe, in reverse order, the stability of the corresponding segments of the motion (in the billiard, through the boundary outwards, in the magnetic field and back again into the billiard through the boundary). This group of four matrices corresponds to a cycle in the Poincaré sections of Fig. 2. (The matrices \mathbf{T} , \mathbf{T}' correspond to the outward and inward passage through the boundary, respectively.)

In our simulations the matrices (1) and the product (2) corresponding to about 25000 cycles were calculated

explicitly, and the Lyapunov exponents, shown in Fig. 4 were computed as the logarithm of the largest eigenvalue (practically, the trace) of the resulting matrix divided by the total time of flight.

The fact that in the $\beta \rightarrow \infty$ limit the inverse magnetic billiard gives back the dynamics of the normal billiard system with elastic walls can be checked also in terms of the stability matrices. A bit lengthy but straightforward calculation yields that if the billiard wall is a circle of curvature q , then

$$\lim_{\beta \rightarrow \infty} (\mathbf{T}(-\beta, -\mu) \mathbf{E}(t, \beta) \mathbf{T}(\beta, \mu)) = - \begin{bmatrix} 1 & 0 \\ -\frac{2q}{\cos \mu} & 1 \end{bmatrix}, \quad (3)$$

which is the stability matrix corresponding to an elastic reflection on the wall of curvature q [13], as it is expected. (The signs of the arguments of \mathbf{T} can be obtained by elementary geometric considerations.)

We now turn to the discussion of the conditions of the experimental realization of the inverse magnetic billiards using GaAs/AlGaAs heterostructure. There are four characteristic lengths in the system: the Fermi wavelength (typically $\lambda_F = 40$ nm [1]), the radius r of the stadium, the cyclotron radius R_c and the mean free path l (which can be as high as 10^4 nm [1]). The classical ballistic motion of the electrons requires that $\lambda_F \ll r, R_c \ll l$. (The last condition assures that the electron travels through several Poincaré cycles without scattering on impurities.) Fig. 4 shows that the relevant values of the ratio r/R_c are in the range of 0.01 – 1.0. The magnetic field can be as high as 2 T without destroying superconductivity. This implies that $R_c \gtrsim 50$ nm (using that the effective mass of electrons $m_{\text{eff}} = 0.067m_e$, where m_e is

the mass of the electron, and $E_F = 14$ meV [1]). Assuming that the size of a superconductor grain is about $r = 1$ μm , the cyclotron radii are 50, 300, 1000 nm corresponding to data R_c/r in Fig. 2. This implies that parameter β in Fig. 2 corresponds to the experimental values of the magnetic field 2, 0.3, 0.2 T, respectively. It is clear that these experimental values do not perfectly fit the condition of the classical motion. The semiclassical or full quantum mechanical treatment of the problem can be an extension of our work.

The advantage of our suggested setup in comparison with Andreev billiards (which is another proposed experimental setup for magnetically tunable chaoticity) is that in our system the electrons travel in a homogeneous heterostructure without any scattering on the boundary of the stadium, whereas in the case of Andreev billiards the normal reflections may suppress the effect as discussed in Ref. 8.

We remark that in a real experiment, the profile of the magnetic field cannot be approximated by a step function as we assumed before. However, the deviation of the magnetic field from the sharp profile can easily be included in classical calculations.

In practice, one would measure the conductance or susceptibility, which should be sensible to the chaotic nature of the system tuned by magnetic field [15].

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- [1] C. W. J. Beenakker and H. van Houten, *Solid State Phys.* **44**, 1 (1991).
 - [2] C. L. Foden, M. L. Leadbeater, J. H. Burroughes, and M. Pepper, *J. Phys.: Condens. Matter* **6**, L127 (1994); M. L. Leadbeater *et al.*, *Phys. Rev. B* **52**, R8629 (1995).
 - [3] M. L. Leadbeater *et al.*, *J. Appl. Phys.* **69**, 4689 (1991); K. M. Krishnan, *Appl. Phys. Lett.* **61**, 2365 (1992).
 - [4] A. K. Geim *et al.*, *Nature (London)* **390**, 259 (1997); A. Smith *et al.*, *Phys. Rev. B* **50**, 14726 (1994).
 - [5] J. E. Müller, *Phys. Rev. Lett.* **68**, 385 (1992); F. M. Peeters and P. Vasilopoulos, *Phys. Rev. B* **47**, 1466 (1993); M. Calvo, *Phys. Rev. B* **48**, 2365 (1993); F. M. Peeters and A. Matulis, *Phys. Rev. B* **48**, 15166 (1993); A. Matulis and F. M. Peeters, *Phys. Rev. Lett.* **72**, 1518 (1994); I. S. Ibrahim and F. M. Peeters, *Phys. Rev. B* **52**, 17321 (1995); H.-S. Sim *et al.*, *Phys. Rev. Lett.* **80**, 1501 (1998); N. Kim, G. Ihm, H.-S. Sim, and K. J. Chang, *Phys. Rev. B* **60**, 8767 (1999); N. Kim, G. Ihm, H.-S. Sim, and T. W. Wang, *Phys. Rev. B* **63**, 235317 (2001); S. M. Badalyan and F. M. Peeters, *Phys. Rev. B* **64**, 155303 (2001); M. Governale and D. Boese, *Appl. Phys. Lett.* **77**, 3215 (2000); J. Reijniers, F. M. Peeters, and A. Matulis, cond-mat/0106368 (unpublished);
 - [6] L. Bunimovich, *Funct. Anal. Appl.* **8**, 73 (1974); L. A. Bunimovich, *Commun. Math. Phys.* **65**, 295 (1979); L. A. Bunimovich, *Commun. Math. Phys.* **130**, 599 (1990).
 - [7] C. W. J. Beenakker and H. van Houten, *Phys. Rev. Lett.* **63**, 1857 (1989); R. Fleischmann, T. Geisel and R. Ketzmerick, *Phys. Rev. Lett.* **68**, 1367 (1992).
 - [8] I. Kosztin, D. L. Maslov and P. M. Goldbart, *Phys. Rev. Lett.* **75**, 1735 (1995).
 - [9] M. V. Berry, *Eur. J. Phys.* **2**, 91 (1981); O. Meplan, F. Brut, and C. Gignoux, *J. Phys. A* **26**, 237 (1993).
 - [10] G. Benettin and J.M. Strelcyn, *Phys. Rev. A* **17**, 773 (1978); O. Biham and M. Kvale, *Phys. Rev. A* **46**, 6334 (1992).
 - [11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, No. 15 in *Interscience Tracts in Pure and Applied Mathematics* (Interscience Publishers, New York, London, Sydney, 1969).
 - [12] T. Tasnádi, *Commun. Math. Phys.* **187**, 597 (1997).
 - [13] T. Tasnádi, *J. Math. Phys.* **37**, 5577 (1996).
 - [14] T. Tasnádi, *J. Math. Phys.* **39**, 3783 (1998).
 - [15] C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P.F. Hopkins, and A. C. Gossard, *Phys. Rev. Lett.* **69**, 506 (1992).